Properties of fuzzy set

Huan Huang

*Department of Mathematics, Jimei University, Xiamen 361021, China*

**Abstract**

In this paper, we give some properties related to platform points of a fuzzy set and their applications.

*Keywords:* Fuzzy set; platform points; Γ-convergence

## 1. Main results

Let $(X,d)$ be a metric space. For $u \in F(X)$, let $[u]_\alpha$ denote the $\alpha$-cut of $u$, i.e.

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0,1], \\ \operatorname{supp} u = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the closure of $S$ in $(X,d)$. $F_{USC}(X)$ is the set of upper semi-continuous fuzzy sets in $X$.

For $u \in F(X)$,

$$D(u) := \{\alpha \in (0,1) : [u]_\alpha \nsubseteq \overline{\{u > \alpha\}}\},$$
$$P(u) := \{\alpha \in (0,1) : \overline{\{u > \alpha\}} \subsetneqq [u]_\alpha\}.$$

$\alpha \in P(u)$ is called a platform point of $u$. Clearly $P(u) \subseteq D(u)$.

We show that $D(u)$ is at most countable when $u \in F(\mathbb{R}^m)$ (Theorem 5.1 of [1]).

In this paper, we show that $D(u)$ is at most countable for fuzzy set $u \in F(X)$ with $([u]_0, d)$ being separable. This result follows directly from the proof of Theorem 5.1 of [1] and the fact that each separable metric space is homeomorphic to a subspace of the Hilbert space $l^2 := \{(x_i)_{i=1}^{+\infty} : \sum_{i=1}^{+\infty} x_i^2 < +\infty\}$. We also give some applications of this conclusion.

*Email address:* hhuangjy@126.com (Huan Huang)

**Theorem 1.1.** *Let $u \in F(l^2)$. Then the set $D(u)$ is at most countable.*

**Proof.** The proof is similar to that of Theorem 5.1 in [1].

A sketch of the proof is given as follows

Similarly as in [1], for $u \in F(l^2)$, $t \in l^2$ and $r \in \mathbb{R}^+$, we can define $S_{u,t,r}(\cdot,\cdot) : \mathbf{S}^1 \times [0,1] \to \{-\infty\} \cup \mathbb{R}$ by

$$S_{u,t,r}(e,\alpha) = \begin{cases} -\infty, & \text{if } [u]_\alpha \cap \overline{B(t,r)} = \emptyset, \\ \sup\{\langle e, x-t\rangle : x \in [u]_\alpha \cap \overline{B(t,r)}\}, & \text{if } [u]_\alpha \cap \overline{B(t,r)} \neq \emptyset, \end{cases}$$

where $\mathbf{S}^1 := \{e \in l^2 : \|e\| = 1\}$ and $\overline{B(t,r)} := \{x \in l^2 : \|x-t\| \leq r\}$.

Similarly as in [1], we can define $D(u,t,r,e)$, which is the discontinuous point of $S_{u,t,r}(e,\cdot)$.

Proceed as in the proof of Lemma A.1. of [1], we can show the conclusion corresponding to Lemma A.1. of [1]: $D(u,t,r) = \cup_{e\in\mathbf{S}^1} D(u,t,r,e)$ is countable.

Note that $2\langle a,b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$ for each $a, b \in l^2$. So proceed as in the proof of Theorem 5.1, we obtain that $D(u)$ is at most countable.

□

**Remark 1.2.** In the proof of Theorem 5.1, the fact that $2\langle a,b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$ for each $a, b \in \mathbb{R}^m$ is used.

Clearly, (A.6) in the proof of Theorem 5.1 can be shown by using the equality $\langle e, x-q\rangle = \frac{\langle y-q, x-q\rangle}{\|y-q\|} = \frac{\|x-q\|^2 + \|y-q\|^2 - \|x-y\|^2}{2\|y-q\|}$.

**Theorem 1.3.** *Let $(X,d)$ be a metric space and $u \in F(X)$. If $([u]_0, d)$ is separable, then the set $D(u)$ is at most countable.*

**Proof.** Let $f$ be a homeomorphism from $([u]_0, d)$ to a subspace of $l^2$.

Consider $u_f \in F(l^2)$ defined by

$$u_f(t) = \begin{cases} u(f^{-1}(t)), & t \in f([u]_0), \\ 0, & t \in l^2 \setminus f([u]_0). \end{cases}$$

Note that $D(u) = D(u_f)$, thus $D(u)$ is at most countable from Theorem 1.1.

□

**Remark 1.4.** Here we mention that the closure operator in the definition of $D(u)$ is taken in $(X,d)$ and the closure operator in the definition of $D(u_f)$ is taken in $l^2$.

**Theorem 1.5.** *Let $(X, d)$ be a metric space and $u \in F(X)$. If $([u]_0, d)$ is separable, then the set $P(u)$ is at most countable.*

**Proof.** The desired result follows immediately from Theorem 1.3 and the fact that $P(u) \subseteq D(u)$.

$\square$

There are various kinds of fuzzy sets [2]. Since the corresponding discussion in [3] is in the framework of normal fuzzy sets, we only discuss normal fuzzy sets in the following.

$$F^1_{USCG}(X) := \{u \in F(X) : [u]_\alpha \in K(X) \text{ for all } \alpha \in (0, 1]\},$$
$$F^1_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in (0, 1]\},$$
$$F_{USC}(X) = \{u \in F(X) : [u]_\alpha \in C(X) \cup \emptyset \text{ for all } \alpha \in (0, 1]\},$$

where $K(X)$ and $C(X)$ denote the set of all non-empty compact subsets of $(X, d)$ and the set of all non-empty closed subsets of $(X, d)$, respectively.

**Theorem 1.6.** *Suppose that $u$, $u_n$, $n = 1, 2, \ldots$, are fuzzy sets in $F^1_{USC}(X)$. If $([u]_0, d)$ is separable, then the following statements are true.*

(i) $u_n \xrightarrow{\Gamma} u$

(ii) $u_n \xrightarrow{a.e.} u\,(K)$.

(iii) $[u]_\alpha = \lim_{n\to\infty}[u_n]_\alpha\,(K)$ *for all* $\alpha \in (0, 1) \setminus P(u)$

(iv) $\lim_{n\to\infty}[u_n]_\alpha\,(K) = [u]_\alpha$ *holds when* $\alpha \in P$*, where* $P$ *is a dense subset of* $(0, 1)\setminus P(u)$.

(v) $\lim_{n\to\infty}[u_n]_\alpha\,(K) = [u]_\alpha$ *holds when* $\alpha \in P$*, where* $P$ *is a countable dense subset of* $(0, 1)\setminus P(u)$.

**Proof.** The desired result follows immediately from Theorem 1.5 and Theorem 3.8 in [3].

$\square$

**Remark 1.7.** Note that each compact metric space is separable. So if $u \in F^1_{USCG}(X)$, then $([u]_0, d)$ is separable. Thus Theorem 1.6 improves Theorem 3.9 in [3].

For $u \in F^1_{USC}(X)$, the set $P_0(u)$ is defined by $P_0(u) := \{\alpha \in (0,1) : \lim_{\beta\to\alpha} H([u]_\beta, [u]_\alpha) \neq 0\}$, where $H$ is the Hausdorff metric on $C(X)$ induced by $d$.

By Theorem 1.5 and Lemma 3.6 in [3], we have that $P_0(u)$ is at most countable for each $u \in F^1_{USCG}(X)$.

Lemma 3.6 in [3] is listed as follows

- Let $U_n \in K(X)$ for $n = 1, 2, \ldots$.
  (i) If $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$, then $U = \bigcap_{n=1}^{+\infty} U_n \in K(X)$ and $H(U_n, U) \to 0$ as $n \to +\infty$.
  (ii) If $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n \subseteq \ldots$ and $V = \overline{\bigcup_{n=1}^{+\infty} U_n} \in K(X)$, then $H(U_n, V) \to 0$ as $n \to +\infty$.

However, for $u \in F^1_{USC}(X)$ with $([u]_0, d)$ being separable, $P_0(u)$ need not be at most countable. A counterexample is given in page 7 of [3].


[1] H. Huang, Characterizations of endograph metric and Γ-convergence on fuzzy sets, Fuzzy Sets Syst. 350 (2018) 55-84

[2] C. Wu, M. Ma, The Basic of Fuzzy Analysis (in Chinese), National Defence Industry press, Beijing, 1991

[3] H. Huang, Properties of several fuzzy set spaces, chinaXiv:202107.00011